Questions:

1. Find all FJ points and KKT points of

Minimize –x3-xy

Subject to x+y=4

x≤2

2. Find the KKT points of a linear program

Max cTx

Subject to Ax≤b

x≥0

and provide comments about these conditions.

3. Find the Lagrangian Dual to the following problem and identify any optimal saddle points.

Minimize x2+y2

Subject to x+y=4

x≤2

X={(0,4), (3,1), (2,2), (1,1)}.

**Solutions**

1 FJ points

u0 | -3x2-y | + u1 | 1 | +v | 1 | = | 0 | (i)

| -x | | 0 | + | 1 | | 0 | (ii)

u1(x-2)=0 (iii)

u1≥0 (iv)

x≤2 (v)

x+y=4 (vi)

Case 1: u1>0

Thus x=2, from the (iii). By (vi) y=2. Using this in (ii) results in v=2. So plugging all of this into (i) results in -14u0+u1+2=0 and so u1=14u0-2 for u0≥1/7. Thus, there are an infinite number of FJ points. They are

x=2, y=2, u0=s, u1=14s-2, v=2 for any s≥0. Their objective values are –12.

Case 2: u1=0. Since y=4-x, (i) becomes u0(-3x2-4+x) +v = 0. From (ii) we get u0x=v. and substituting this into the new (i), we get u0(-3x2-4+x) + u0x = 0. This simplifies to u0(-3x2-4+2x) = 0 So if u0=0, then v=0 and this is not a FJ point since all of the lagrange multipliers are 0. Therefore. (-3x2-4+2x) = 0. Using the quadratic formula results in (-2 ± (4-24)1/2)/4. Since this is an imaginary number, there are no FJ points in this branch.

1. The KKT conditions are

| -3x2-y | + u | 1 | +v | 1 | = | 0 | (i)

| -x | | 0 | + | 1 | | 0 | (ii)

u(x-2)=0 (iii)

u≥0 (iv)

x≤2 (v)

x+y=4 (vi)

Case 1: u>0

Thus x=2, from the (iii). By (vi) y=2. Using this in (ii) results in v=2. So plugging all of this into (i) results in -14-2+u+2=0 and so u=12.

There is a KKT point at x=2, y=2, u=12, v=2. It’s objective value is –12.

Case 2: u=0.

So (i) becomes -3x2-y +v = 0. From (ii) we get x=v, and substituting this into the new (i), we get -3x2-y +x = 0. From (vi) we get y=4-x. Substituting this into the most recent version of (i), -3x2-4+x +x = 0. So -3x2+2x-4 = 0. Using the quadratic formula results in(-2 ± (4-48)1/2)/6. Since this is an imaginary number, there are no KKT points with u=0.

The only KKT point is at x=2, y=2, u=12, v=2. It’s objective value is –12.

2. Find the KKT points of a linear program

Max cTx

Subject to Ax≤b

x≥0

and provide comments about these conditions.

The KKT conditions are

c+uTA –vT=0

ui(Σj=1naijxj-bi)=0 for all i=1,…,m

ui≥0 for all i=1,…,m

vi≥0 for all i=1,…,n

vixi=0 for all i=1,…,n

Ax≤b

x≥0

Comments

Ax≤b, x≥0 is just the x being primal feasible.

c+uTA –vT=0, can be written as

uTA ≥c since v≥0. Futhermore, u≥0, so these three conditions imply dual feasiblility.

Finally the v’s represent the slack in the dual constraints and so vixi=0 for all i=1,…,n is half of complimentary slackness. The other half of complimentary slackness is given from ui(Σj=1naijxj-bi)=0 for all i=1,…,m. Thus the KKT points of a LP are primal feasible, dual feasible and complimentary slackness and are thus optimal solutions.

3. The Lagrangian dual of

Minimize x2+y2

Subject to x+y=4

x≤2

is

If Θ (u,v) = inf{x2+y2 +v(x+y-4) + u(x-2)}

subject to u≥0

and (x,y) є X={(0,4), (3,1), (2,2), (1,1)}.

It is easy to see that the optimal solution to the primal occurs at x=2, y=2. There are 4 possible values for (x,y). Plugging each one into the dual results in the following

Sup{ 16 +v(0) + u(-2), 10+v(0) +u(1), 8+v(0)+u(0), 5 + v(-1)+u(-1) }

u≥0

This reduces to max {16-2u, 10 +u, 8, 5-v-u}, since u≥0, the second case dominates by the 3rd case. So max {16-2u, 10+u, 5-v-u} Running through conditions we get 16-2u as long as 16-2u≤10+u and 5-v-u. Solving this logic leads to the following dual function.

If Θ(u,v) = 16-2u if u≤2 and u≤11+v

10+u if u>2 and u≥(-5-v)/2

5-v-u else

And Φ(x,u,v) = x2+y2 +v(x+y-4) + u(x-2).

Since the optimal primal solution is clearly (2,2) we can plug that in and get.

\_

Φ(x,u,v) = 22+22 +v(2+2-4) + u(2-2) = 8.

Observe that (u.v) can now be anything and so any values of (u,v) are a Lagrangian Saddle point of optimality.